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1989 J. Phys. A: Math. Gen. 22 3103

(http://iopscience.iop.org/0305-4470/22/15/024)

J. Phys. A: Math. Gen. 22 (1989) 3103-3112. Printed in the UK

String-like lattice models and Hecke algebras

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Received 12 July 1988

Abstract. We discuss lattice statistical mechanical models with n-dimensional (static endpoint) string-like degrees of freedom. In the infinite string tension limit these models give representations of Hecke algebras $H_k(q)$, quotient algebras of which may be used to construct Virasoro algebras in the limit $k\to\infty$ and when $q=4\cos^2(\pi/r)$ ($r\in\mathbb{Z}$). We give the regular representation and well defined representations associated with and containing each irreducible representation, including non-unitarisable cases. We use these models to obtain a complete set of primitive idempotents, and hence the central idempotents, of the generic algebra.

1. Introduction

The two-dimensional statistical mechanical models solved by Andrews *et al* (1984), and found to have critical exponents coincident with those of conformal field theories (Friedan *et al* 1984, Huse 1984), may be constructed directly at criticality in terms of Temperley-Lieb algebras (Temperley and Lieb 1971, Kuniba *et al* 1986). In a recent paper (Martin 1987) we showed how the models of Andrews *et al* (1984) can in turn be used to find the irreducible representations of such algebras.

These algebras are expected to be related to the algebra of conformal transformations (Kuniba *et al* 1986, Belavin *et al* 1984, Shultz *et al* 1964). More generally the A_n Hecke algebra $H_k(q)$ with k generators U_i obeying

$$\begin{aligned} &U_i^2=\sqrt{q}\,U_i \\ &U_iU_{i+1}U_i-U_i=U_{i+1}U_iU_{i+1}-U_{i+1} \\ &[U_i,U_j]=0 \qquad |i-j|\neq 1 \end{aligned} \tag{1}$$

with q a scalar parameter (Hoefsmit 1974), is also important in solvable statistical mechanical models (Date *et al* 1987). The Temperley-Lieb algebra is obtained from $H_k(q)$ by imposing the additional relations

$$U_iU_{i+1}U_i-U_i=0. \tag{2}$$

The algebras then associated with conformal theories have

$$q=4\cos^2(\pi/r) \qquad r=1,2,3,\ldots. \tag{3}$$

For example, $H_k(2)$ has a quotient algebra obeying

$$1-[1/(q-1)][\sqrt{q}(U_i+U_{i+1})-U_iU_{i+1}-U_{i+1}U_i]=0. \tag{4}$$

With $V_i=1-2U_i/\sqrt{q}$ we then find that the objects

$$c_i=\frac{1}{2}\left(\prod_{j=1}^{4j-1<2i+2}V_{4j-1}+\prod_{j=0}^{4j+1<2i+2}V_{4j+1}\right)V_{2i+2} \qquad i>0 \tag{5}$$

with

$$c_{-i} = c_i^{\mathrm{T}} \qquad \text{and} \qquad c_0 = V_2/\sqrt{2} \tag{6, 7}$$

obey fermion anticommutation relations $\{c_i, c_j\} = \delta_{i,-j}$. It is well known how to construct representations of the Virasoro algebra with central charge $c = \frac{1}{2}$ from such objects (see Belavin *et al* 1988).

Existing statistical mechanical models constructed using the Hecke algebras (see Date *et al* 1987, Pasquier 1988) are of rather uncertain physical content, and are too inflexible to describe all representations, while the abstract recipes of Wenzl (1988) and Hoefsmit are restricted to unitarisable or generic cases. Here we discuss models with n-dimensional static endpoint string-like degrees of freedom which, on restriction to the infinite string tension limit, give the irreducible representations of $H_k(q)$ in a straightforward way. We write down the regular representation and well defined representations associated with and containing each irreducible representation, for all r. We give a framework for the discussion of more general representations. This enables us to exhibit the structure of the generic algebra ($r \notin \mathbb{Z}$) as a direct sum of matrix algebras, to write down a set of primitive idempotents, and to give the central idempotents of the algebra. Since the transfer matrix, which is a statistical mechanical antecedent of the stress-energy tensor (Belavin *et al* 1988), is written in terms of the operators $\{U_i\}$, this analysis is hopefully the precursor to revealing a physically sensible action for the Virasoro algebra in the statistical mechanical context. For example, the Temperley-Lieb algebra unitarity is that the unique primitive central idempotent of $H_{r-2}(q)$ vanishes for any $r-2$ adjacent operators (i.e. equation (4) in the case $q = 2$).

2. The model

Consider a string in $n+1$ Euclidean dimensions which has static endpoints at $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, with $\boldsymbol{b} - \boldsymbol{a} = \boldsymbol{\alpha}$ given by $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, in some rectilinear coordinate system in the first n dimensions (we put $\alpha_i \geqslant 0$ and order so that $\alpha_i > \alpha_j \Rightarrow i < j$ without loss of generality). Now replace the n-dimensional subspace with an n-dimensional hypercubic lattice. The endpoints sit on sites, so that the displacement coordinates α_i become integers and the string is replaced by a sequence of lattice points, ordered by a discretised arc length m, with the 'stringiness' property that the points m and $m+1$ are nearest neighbours.

The (Euclidean) time direction is also discretised. The configuration space of the string at a given instant may then be represented on a 'forward moving' path (see figure 1) through a two-dimensional square lattice of length equal to the total arc length m_τ (which will be larger than $m_\infty = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ in general). The two-dimensional lattice site variables take values from the possible coordinates of lattice sites in the n-dimensional space. As such they are constrained to respect the nearest-neighbour condition, which is then enforced for all paths through the 2D lattice. A path adjacent on the right (say) to our original path gives the immediate discretised time evolution of the string. There are many distinct forward moving paths on the 2D lattice which could represent the instantaneous string. We will see that these are equivalent, since the corresponding distinct transfer matrices (Kogut 1979) for time evolution of the string are related by similarity transformations.

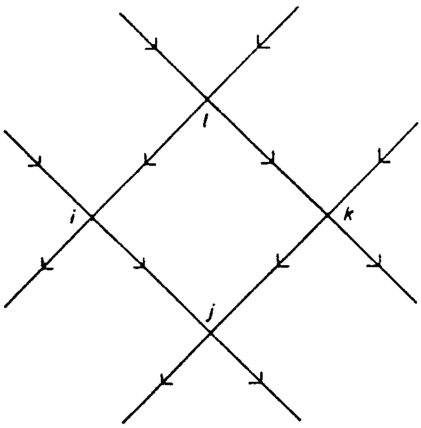

Figure 1. The forward direction on the two-dimensional lattice.

What we have done is to pull the world sheet of the string, arbitrarily convoluted in n dimensions, flat—but to encode the convolutions in the 2D lattice variables. The 'depth' of the 2D lattice varies depending on the number of steps in the string m_τ, but the nearest-neighbour rule decouples the configuration space of the world sheet into sectors of constant depth. To see this consider the possible string configurations accessible from a given configuration by the nearest-neighbour rule.

This means that we can break up the configuration space into subspaces giving successive contributions to a high string tension expansion. Here we will restrict attention to the infinite string tension limit, $\tau=\infty$. The depth of the lattice is then fixed at m_∞ and we may effectively redefine n so that $\alpha_i>0$. In the limit we can also associate an 'internal' field with the links of the string (we will consider a scalar field). We write $y(i)\in\mathbb{R}$ for the scalar on the link between arc length $i-1$ and i. Thus to define an instantaneous configuration of a string it is necessary to specify the path it describes in n-dimensional space and the value of $y(i)$ on each link. These values may be chosen arbitrarily at each link, but the interactions we will consider below are such that configuration space will decouple into regions with a fixed number of links carrying any particular scalar value.

We define a partition function for the 2D lattice system by

$$Z=\sum_{\text{configurations}}\exp\left(\sum_{\text{plaquettes } ijkl}\ln(W_{ijkl}(\beta))+\text{other interactions}\right) \tag{8}$$

where the argument is a classical Hamiltonian dependent on the configuration of the world sheet and interaction parameters represented generically by β. The plaquette weights $W_{ijkl}(\beta)$ depend on: (i) the 2D lattice site variables $x(i)$, $x(j)$, $x(k)$ and $x(l)$ (see figure 1) which we have taken to run anticlockwise from the left (the direction of earlier time) round the plaquette $ijkl$ to the top (lesser arc length); and (ii) the associated link variables. Each site variable, $x(i)=(x_1,x_2,\ldots,x_n)$, gives a point on the n-dimensional lattice. We will refer to the arc length common to the ith and kth sites as the arc length of the plaquette. We may then think of the interaction at arc length i as simply associating distinct weights with time evolutions which interchange or do not interchange the n-dimensional string links common to site i. If we restrict attention to these plaquette interactions then we may write down a $\tau=\infty$ transfer matrix T, such that $Z = \langle T^N \rangle$ where N is the number of time steps and the brackets represent some initial and final states of the string. This transfer matrix is

$$T = \left(\prod_{i=1}^{[m_\infty/2]} \mathfrak{W}_{2i-1} \right) \left(\prod_{i=1}^{[(m_\infty-1)/2]} \mathfrak{W}_{2i} \right) = S T' S^{-1} = S \left(\prod_{i=1}^{m_\infty - 1} \mathfrak{W}_i \right) S^{-1}. \tag{9}$$

Here T' represents a different forward-moving path on the lattice with

$$S = \prod_{i=1}^{[(m_\infty-2)/2]} \left(\prod_{j=1}^{2i+a-1} \mathfrak{W}_{m_\infty - a - 2i + j} \right) \tag{10}$$

($a = 0, 1$ for m_∞ even, odd), cf the corner transfer matrix of Baxter (1982). The *single plaquette* transfer matrix $\mathfrak{W}_i$ is

$$(\mathfrak{W}_i)_{jk} = \left(\prod_{l \neq i} \delta_{x_j(l), x_k(l)} \right) W_{(x_j(i), x_j(i+1), x_k(i), x_j(i-1))}(\beta) \tag{11}$$

where $x_j(i)$ denotes the position at arc length i of the string in configuration j. We introduce U_i by

$$\mathfrak{W}_i \sim (1 + l(\beta) U_i). \tag{12}$$

The distinct types of plaquette configuration allowed by the nearest-neighbour rule are

$$\begin{aligned} &(x(i), x(j), x(k), x(l)) \\ &\quad = (x + e_a, x + e_a + e_c, x + e_b, x) \qquad \text{with } a = b = c && \text{(c1)} \\ &\quad = (x + e_a, x + e_a + e_c, x + e_b, x) \qquad \text{with } a = b \neq c && \text{(c2)} \\ &\quad = (x + e_a, x + e_a + e_c, x + e_b, x) \qquad \text{with } b = c && \text{(c3)} \end{aligned}$$

where x is some point on the n-dimensional lattice (in the box $0 \leqslant x_i - a_i \leqslant \alpha_i$ for all $i = 1, \ldots, n$) and e_i is the unit vector in the ith direction. In (c1) and (c2) the link variables are unchanged; in (c3) distinct link variables are exchanged between opposing links, and/or $a \neq b$. Note that the first and second configurations do not involve any *movement* of the string.

It is convenient to introduce the notation

$$x \equiv x. = [x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n] \qquad x._n = 0 \tag{13}$$

and consider the case $\alpha_i =$ constant. Here $x.$ is the projection of the string into the $(n-1)$-dimensional space orthogonal to the straight line α between the endpoints a and b. In this framework the initial *and* final coordinates are zero, but the $x._i$ basis is not orthogonal. We note that $x_a - x_c$ is then the projected distance of the string from α in the e_a-e_c plane. The kind of lattice system we want to embrace would model a continuum action dependent on the area of the world sheet in some region of its parameter space. The lattice model could, for example, be *weighted* to favour the small region of configuration space in which the string paths keep close to α (a consequence of string tension in the continuum), while entropy considerations would then promote the contributions of more energetically typical paths.

The weights we use have such ground-state properties for small values of their interaction parameters (and $n \leqslant r - 1$), and are of the following form (cf Kuniba *et al* 1986):

$$W_{c1} = 1 \tag{14}$$

$$W_{c2} = [f^2(\chi + \beta r/\pi) f^2(1) / f^2(\chi) f^2(1 - \beta r/\pi)] \tag{15}$$

$$W_{c3} = l(\beta)[f(\chi + 1) f(\chi - 1) / f^2(\chi)]. \tag{16}$$

where $\chi = x_n - x_{\cdot} + y(i) - y(i+1)$ for the plaquette at arc length i. Consider

$$f(x) = [\sin^{1/2}(x\pi/r)\exp(-h^2\cos(x\pi/r))] \tag{17}$$

and

$$l(\beta) = f^2(\beta r/\pi) f^{-2}(1-\beta r/\pi) \tag{18}$$

where h and β are the interaction parameters (we will use the notation f_0 for $f|_{h=0}$). Although these weights look complicated they have the merit that they satisfy the star-triangle relation (Baxter 1982) at zeroth and first order in h^2. The zeroth-order relation implies (cf Kuniba *et al* 1986) that the following operators (from (12)) obey the Hecke algebra relations:

$$(U_i)_{jk} = \begin{cases} [f_0^2(\chi+1)/f_0^2(\chi)] & \text{in case (c2) at arc length } i \\ [f_0(\chi+1)f_0(\chi-1)/f_0^2(\chi)] & \text{in case (c3)} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

(cf Hoefsmit 1974). We will not pursue any consequences of the first-order relation here, as there remain a number of interesting results to be noted from the zeroth order. These are largely technical, but can nonetheless contribute to a physical interpretation of the role of Hecke algebras in statistical mechanics. This is because the exercise of drawing convincing parallels between the infinite-dimensional limits of various multi-matrix algebras, the unitarisable quotients of $H_k(q)$, and the Virasoro algebras (as exemplified in the introduction) may be assisted by a knowledge of the full algebraic structure which lies behind these quotients.

For example, with $y(i) = \text{constant}$, strings from $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ to $\boldsymbol{b} = (a_1, a_2, \ldots, a_i+1, \ldots, a_j+1, \ldots, a_k+1, \ldots)$ with $a_i - a_j = x$; $a_i - a_k = y$; and $a_j - a_k = y - x$; where $\{a_l \in \mathbb{C}$ for all $l \in \{1, \ldots, n\}\}$, give rise, after an obvious but noteworthy basis change, to the 'alternative' representation

$$U_1 = \begin{bmatrix} X & X(\sqrt{q}-X) & & & & \\ 1 & (\sqrt{q}-X) & & & & \\ & & Y & Y(\sqrt{q}-Y) & & \\ & & 1 & (\sqrt{q}-Y) & & \\ & & & & Z & Z(\sqrt{q}-Z) \\ & & & & 1 & (\sqrt{q}-Z) \end{bmatrix}$$

$$U_2 = \begin{bmatrix} X & & X(\sqrt{q}-X) & & & \\ & Y & & & Y(\sqrt{q}-Y) & \\ 1 & & (\sqrt{q}-X) & & & \\ & & & Z & & Z(\sqrt{q}-Z) \\ & 1 & & & (\sqrt{q}-Y) & \\ & & & 1 & & (\sqrt{q}-Z) \end{bmatrix} \tag{20}$$

where $X = s(x+1)/s(x)$, $Y = s(y+1)/s(y)$, $Z = s(y-x+1)/s(y-x)$ and $s(x) = \sin(x\pi/r)$. These matrices may be seen to satisfy the relations for *all* x, y, $(x-y) \neq 0$ on repeated application of the basic trigonometric identities. In general, changes in the position of the endpoint $\boldsymbol{a}$ while holding $\boldsymbol{\alpha}$ fixed correspond, by continuity, to similarity transformations. If some $(a_l - a_m) \to n_{lm} \in \mathbb{Q}$ for $l, m \in i, j, k$ these may be singular when $r \in \mathbb{Z}$. The dimensions of the inequivalent representations labelled by distinct lattice vectors $\boldsymbol{\alpha}$ such that $\alpha_i > 0$ and $\alpha_i > \alpha_j \Rightarrow j > i$ is

$$D_\alpha = m_\infty! \left(\prod_{i=1}^{n} (\alpha_i !) \right)^{-1} .$$

When $q = 4$ these correspond to permutation representations (Robinson 1961) of the permutation group S_{m_∞} (see later). We will refer to them as such for all q.

An important technical point here is that the representation exemplified by (20) is not necessarily isomorphic to the original one (equation (19)) when, for instance, $X = 0$. In such cases the apparent isomorphism class of representations generated by varying x (i.e. generated from (19) by moving $\boldsymbol{a}$) may break up into various disjoint classes. This is a consequence of working with fields which depend on q, which is not a problem provided some care is exercised. The alternative representation with $\alpha_i = 1$ $(i = 1, n)$ is the regular representation for $H_{n-1}(q)$. This is not true of the original representation (19) in general unless the algebra is semisimple (i.e. $r \notin \mathbb{Z}$ or $r \in \mathbb{Z} > n$, see later). The non-semisimple cases with $n = 3$, for example, are $H_2(0)$ and $H_2(1)$ for which the regular representations may then be written as direct sums of indecomposables as follows

$q = 0$:

$$U_1 = \begin{pmatrix} 0 &&&&& \\ 1 & 0 &&&& \\ && 0 &&& \\ && 1 & 0 && \\ &&&& 0 & \\ &&&& 1 & 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 0 & 1 &&&& \\ & 0 &&&& \\ && 0 & 1 && \\ &&& 0 && \\ &&&& 0 & \\ &&&& 1 & 0 \end{pmatrix}$$

$q = 1$:

$$U_1 = \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 1 & 1 &&& \\ & 0 & 1 &&& \\ && 1 &&& \\ &&& 1 & 0 & 1 \\ &&& 1 & 0 & 1 \\ &&& 0 & 0 & 0 \end{pmatrix} .$$

These are isomorphic to (20) for all x, y but not, for example, to (19) with $x = y - x = -\frac{1}{2}$ $(q = 0)$ or $= -1$ $(q = 1)$.

If the field $y(i)$ is not constant then, in addition to $\boldsymbol{\alpha}$ being a constant within a basis, the vectors $\boldsymbol{\alpha}_y$, which are the sums of vectors associated with links carrying a common value of $y(i) = y$, are constant. The set $\{\boldsymbol{\alpha}_y;$ distinct values of y in the field$\}$ then label the representation (see later).

In the basis $a_l = -l$ (for all l) considerable simplification occurs provided r is not integer. Firstly, the 'standard' subspace of string configurations for which $x_i + i > x_j + j$ implies $i < j$ for all $\boldsymbol{x}$ on the string decouples from the rest (consider equation (19)). The resultant representations for $H_k(q)$ when $y(i) =$ constant are a complete set of irreducibles corresponding to those in Wenzl (1988). When $q = 4$ the objects

$$t_i = 1 - \exp(\theta)\, U_i = \mathfrak{W}_i|_{h=0, l(\beta)=-\exp(\theta)} \tag{21}$$

($\exp(\theta)+\exp(-\theta)=\sqrt{q}$) generate S_{k+1} (see equation (1)), and provided r is not integer the algebraic structure of $H_k(q)$ is isomorphic to the S_{k+1} group algebra. Now the number of string configurations or 'walks' in each basis (labelled by $\boldsymbol{\alpha}$) can be determined as follows. The number of ways of putting n distinguishable steps into a (thus numbered) walk of length n is $n!$. On the other hand for each walk allowed in our candidate basis for an irreducible representation (i.e. with $x_i+i>x_j+j \Rightarrow i<j$) we can generate some number, say P_α, of distinct numbered walks. First we number the steps of the allowed walk. Then we generate new walks by interchanging the first step in turn with each other step in the same direction and the first step in each higher indexed dimension. Take the set of numbered walks thus produced and generate new walks by iterating this procedure. At the nth stage this means permuting the nth numbered step with each later numbered step in the same direction and the nth numbered step in each higher indexed dimension. Then P_α is the product of the numbers of such interchanges at each stage. These in turn are equivalent to hooks on a corresponding Young tableau (Hamermesh 1962). The procedure generates all numbered walks once so the dimension of the irreducible representation labelled by $\boldsymbol{\alpha}$ is $n!/P_\alpha$ as required.

The number of copies of the irreducible associated with $\boldsymbol{\alpha}'$ (say) in the permutation representation $\boldsymbol{\alpha}$ is given by the number of allowed walks over $\boldsymbol{\alpha}'$ in which the first α_1 steps are in the same direction, the next α_2 are in the same direction and so on (by continuity with $q=4$ and then from Hamermesh (1962)). For example the number of copies of the irreducible associated with $\boldsymbol{\alpha}$ itself is 1.

If $r\in\mathbb{Z}$ we may still uniquely associate each irreducible with an $\boldsymbol{\alpha}$. The dimensions are less than or equal to the generic dimensions (and possibly zero). Firstly, if $\alpha_1-\alpha_n<r-n+1$, then as r is tuned to an integer the subspace of basis configurations (within the generic irreducible) with $x_{\cdot 1}=x_1-x_n<r$ for all $\boldsymbol{x}$ decouples from the rest and gives a unitary representation (Wenzl 1988). Some matrix elements outside the subspace would then be divergent (see equation (19)). As in the Temperley-Lieb case these can be controlled by taking $r\to r+\varepsilon$, and making certain similarity transformations before allowing $\varepsilon\to 0$ (see Martin (1988) and note, for example, that $\mathrm{Tr}(U_i)=j\sqrt{q}$ with $j\in\mathbb{Z}$ in any representation); however, unitarity is lost. The representation will then be reducible (although indecomposable) in general, like the sub-blocks of the regular representation in our example. In general the new irreducibles themselves require basis changes (as well as restricting to subspaces which are generalisations of those discussed in Martin 1988 for the Temperley-Lieb case) before $\varepsilon\to 0$. Of course the permutation representations are already defined for all q, what we have done is to show that they contain the appropriate irreducibles.

When $y(i)$ is not constant, r not integer and $a_l=-l$, in the standard subspace the corresponding representations are continuous with the outer product representations of S_{k+1}, denoted $\boldsymbol{\alpha}_{y_1}\cdot\boldsymbol{\alpha}_{y_2}\cdot\boldsymbol{\alpha}_{y_3}\ldots$ where the product is over all distinct values of $y(i)$ (Robinson 1961). The dimensions are given inductively by

$$\dim(\boldsymbol{\alpha}_{y_1}\cdot\boldsymbol{\alpha}_{y_2}\cdot\boldsymbol{\alpha}_{y_3}\ldots)=\binom{(k+1)}{|\boldsymbol{\alpha}_{y_1}|}\frac{|\boldsymbol{\alpha}_{y_1}|!}{P_1}\dim(\boldsymbol{\alpha}_{y_2}\cdot\boldsymbol{\alpha}_{y_3}\ldots)$$

where $|\boldsymbol{\alpha}_y|$ is the number of steps in $\boldsymbol{\alpha}_y$, so that $(k+1)=\Sigma_i|\boldsymbol{\alpha}_{y_i}|$, and P_1 is the product of hook lengths P_α with $\boldsymbol{\alpha}=\boldsymbol{\alpha}_{y_1}$. In particular if $y(i)$ takes only two distinct values (y,x) then we can construct a diagonal t_0 such that

$$t_0^2=f(y-x)t_0+g(y-x)1$$

where f and g are some functions and

$$t_0 t_1 t_0 t_1 = t_1 t_0 t_1 t_0$$

and hence give a complete set of irreducibles for the generic B_n and D_n type Hecke algebras (see Hoefsmit 1974). The advantage of the present construction is that if $a_l \neq -l$ it works perfectly well with the permutation representations for all r. The dimensions (although not necessarily the decompositions) are given by continuity and the associativity of outer product and direct sum when $q = 4$.

The interest in the outer product representations arises from the fact that when $q = 4$ they are induced using $S_{p+q} \supset S_p \times S_q$. As we will discuss in a subsequent paper, this procedure can act for the transfer matrix like a block-spin renormalisation.

We now construct primitive idempotents. These are of more than just aesthetic interest; they enable us to make purely algebraic calculations for the largest eigenvalues of the transfer matrix in each irreducible subspace (see, for example, Baxter 1982).

For each distinct $\boldsymbol{\alpha}$ consider the unique string configuration o passing through the points

$$(0, \ldots, 0), (0, \ldots, 0, \alpha_n), (0, \ldots, 0, \alpha_{n-1}, \alpha_n), \ldots, (0, \alpha_2, \ldots, \alpha_n), (\alpha_1, \ldots, \alpha_n)$$

and associate with this configuration the idempotent

$$^{\alpha}E_{\mathrm{o}} = E^{1}_{\alpha_n - 1} E^{\alpha_n + 1}_{\alpha_{n-1} - 1} E^{\alpha_n + \alpha_{n-1} + 1}_{\alpha_{n-2} - 1} \ldots E^{\alpha_n + \ldots + \alpha_2 + 1}_{\alpha_1 - 1} \tag{22}$$

where E^b_c is constructed in the same way as the idempotent defined in Martin (1988) (for the Temperley-Lieb algebra) as $\mathrm{Idem}_{(b-1)/2}[c+1]$. Namely, $E^b_0 = 1$,

$$E^1_c = E^1_{c-1}(1 - \kappa_c U_c) E^1_{c-1}$$

where $1/\kappa_c = \sqrt{q} - \kappa_{c-1}$ with $\kappa_1 = 1/\sqrt{q}$, i.e.

$$\kappa_c = \sin(c\pi/r)/\sin((c+1)\pi/r)$$

and $E^b_c = E^1_c\,(U_i \to U_{i+b-1}$ for all $i)$, so that $E^b_1 = 1 - q^{-1/2} U_b$, and so on. The idempotent has analogous properties in the Hecke case, i.e.

$$U_i E^b_c = 0 \qquad \text{if } b \leq i < b + c.$$

The proof of this, and of idempotency, is by straightforward induction using the Hecke relations and the definition of κ_c. Returning to (22), for example if $\alpha_i = 1$ $(i = 1, \ldots, n)$ then $^{\alpha}E_0 = 1$. Note that in general the structure of this idempotent is a product of the idempotents associated with the trivial representations in various commuting subalgebras. In what follows this simply reaffirms the analogy with permutation representations of the permutation group (cf Robinson 1961).

Now further associate with each configuration t an operator L_t obtained from $E_{\mathrm{o}}(= L_{\mathrm{o}})$ by repeated use of

$$L_t = \left(\frac{\sin((y-1)\pi/r)}{\sin((y+1)\pi/r)}\right)^{1/2} \left(1 - \frac{\sin((y)\pi/r)}{\sin((y-1)\pi/r)}\, U_i\right) L_s \tag{23}$$

where configurations t and s differ only in the ith position and these positions differ in only the jth and kth coordinates with

$$y = (x_t(i-1))_j - x_t(i-1))_k$$

and $x_t(i)_j = x_t(i-1)_j + 1$ if $j < k$ (this latter defines a partial order on configurations with o the unique lowest). Every L depends on $\boldsymbol{a}$ and $\boldsymbol{b}$, but $E_o = L_o$ depends only on $\boldsymbol{b} - \boldsymbol{a} = \boldsymbol{\alpha}$. Note that if t is of the form

$$\ldots, \boldsymbol{x}_t(i-1) = \boldsymbol{x}, \boldsymbol{x}_t(i) = \boldsymbol{x} + \boldsymbol{e}_a, \boldsymbol{x}_t(i+1) = \boldsymbol{x} + 2\boldsymbol{e}_a, \ldots$$

then $U_i L_t = 0$. The proof is straightforward.

The left ideal generated from E_o thus terminates with the element L_f where f is the unique final configuration, passing through the points

$$(0, \ldots, 0), (\alpha_1, 0, \ldots, 0), (\alpha_1, \alpha_2, 0, \ldots, 0), \ldots, (\alpha_1, \ldots, \alpha_{n-1}, 0), (\alpha_1, \ldots, \alpha_n).$$

Note that in general such an ideal is well defined for all r.

It is possible to choose $\boldsymbol{a}$ such that some of the operators are not defined. However (defining $R_t = L_t^{\mathrm{T}}$ where T means writing the operators in reverse order) the operator $E_{I^\alpha} = k_{I^\alpha} L_{I^\alpha} R_{I^\alpha}$, where k_{I^α} is just some $\boldsymbol{a}$-dependent normalisation and I^α is the unique lowest configuration for a given $\boldsymbol{\alpha}$ for which $x_i - a_i > x_j - a_j$ implies $i<j$ for all $\boldsymbol{x}$ on the string, is well defined when $a_l = -l$, for all $r \notin \mathbb{Z}$, and for *all* r provided $\alpha_1 - \alpha_n < r - n + 1$. We now restrict ourselves to this basis.

We find $E_{I^\alpha} E_{I^\gamma} = \delta_{\alpha,\gamma} E_{I^\alpha}$ (see later). Given this it is easy to prove that for s, t higher than or equal to I^α, I^γ respectively (i.e. obtained from them in the sense of (23)) we have

$$R_s L_t = \delta_{s,t} E_{I^\alpha}. \tag{24}$$

A complete set of primitive idempotents are thus $I_s = L_s R_s$ for each s obtained from each I^α, and the minimal central idempotents are

$$C_\alpha = \sum_{\substack{\text{configurations} \\ \text{obtained from } I^\alpha : s}} L_s R_s \tag{25}$$

for each $\boldsymbol{\alpha}$.

The crucial properties of E_{I^α} come from the fact that it takes the form $F_\alpha L_t R_t F_\alpha$ where

$$F_\alpha = \prod_{j=2}^{n} \left(\prod_{i=1}^{(\alpha_j - \sum_{k=j+1}^{n} \alpha_k)} F_{j-1}^{1+(i-1)j} \right) \tag{26}$$

t is the unique highest walk for which $x_i - a_i > x_j - a_j$ implies $i > j$ for all $\boldsymbol{x}$ on the string and F_c^b is obtained by making the replacement $U_i \to \sqrt{q} - U_i$ for all U_i in E_c^b (so $U_i F_c^b = \sqrt{q}\, F_c^b$ if $b \leqslant i < b + c$). The first few of these are

$$F_0^1 = 1 \qquad F_1^1 = q^{-1/2} U_1 \qquad F_2^1 = \frac{1}{\sqrt{q}(q-1)} (U_1 U_2 U_1 - U_1) \ldots.$$

With four operators, for example, the $\{E_{I^\alpha}\}$ are then (up to normalisation):

$\boldsymbol{\alpha} = (5)$: E_4^1 $\qquad$ $\boldsymbol{\alpha} = (4, 1)$: $F_1^1 E_3^2 F_1^1$

$\boldsymbol{\alpha} = (3, 2)$: $F_1^1 F_1^3 U_2 E_1^1 E_2^3 U_2 F_1^3 F_1^1$

$\boldsymbol{\alpha} = (3, 1, 1)$: $F_2^1 E_2^3 F_2^1$ $\qquad$ $\boldsymbol{\alpha} = (2, 2, 1)$: $F_2^1 F_1^4 U_3 E_1^2 E_1^4 U_3 F_1^4 F_2^1$

$\boldsymbol{\alpha} = (2, 1, 1, 1)$: $F_3^1 E_1^4 F_3^1$ $\qquad$ $\boldsymbol{\alpha} = (1, 1, 1, 1, 1)$: F_4^1.

Note that this construction is not equivalent to that of Wenzl (1988). There is no unique minimal word set in terms of generators $\{U_i\}$ or $\{t_i\}$ for $H_k(q)$ (consider (1)), so the two approaches give complementary expressions for the same set of idempotents. In fact neither set is ideally built to exhibit the q dependence of central idempotents. Work in this area continues. A central idempotent will often be defined at q values for which the basis dependent primitive idempotents are not. This could suggest suitable bases for the non-unitarisable irreducibles.

Acknowledgments

I would like to thank B Westbury for invaluable conversations, and the SERC for financial support in part of this work.

Note added in proof. Primitive idempotents for $H_k(q)$ are also constructed in Dipper and James (1987 *Proc. London Math. Soc.* **54** 57).

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